

Quantum Tetrahedra

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Abstract

We discuss in details the role of Wigner $6j$ symbol as the basic building block unifying such different fields as state sum models for quantum geometry, topological quantum field theory, statistical lattice models and quantum computing. The apparent twofold nature of the $6j$ symbol displayed in quantum field theory and quantum computing –a quantum tetrahedron and a computational gate– is shown to merge together in a unified quantum–computational $SU(2)$ –state sum framework.

Keywords: quantum theory of angular momentum; Wigner $6j$ symbol; discretized quantum gravity; spin network quantum simulator

1 Introduction



The above illustration shows a variant woodcut printer's device on verso last leaf of a rare XVI century edition of Plato's *Timaeus*, (*Divini Platonis Operum a Marsilio Ficino tralatorum, Tomus Quartus. Lugduni, apud Joan Tornaesium M.D.XXXXX*). The printer's device to the colophon shows a medaillon with a tetrahedron in centre, and the motto round the border: *Nescit Labi Virtus*, Virtue cannot fail¹. This woodcut beautifully illustrates the role of the perfect shape of the tetrahedron in classical culture. The tetrahedron conveys such an impression of strong stability as to be considered as an epitome of virtue, unfailingly capturing us with the depth and elegance of its shape. However, as comfortable as it may seem, this time-honored geometrical shape smuggles energy into some of the more conservative aspects of Mathematics, Physics and Chemistry, since it is perceptive of where the truth hides away from us: the quantum world. As Enzo says, the geometry of the tetrahedron actually takes us on a trip pointing to unexpected connections between the classical and the quantum. He has indeed often entertained us with descriptions of open terrains of Physics and Chemistry which are bumpy, filled with chemical bonds and polyhedra, and which bend abruptly in unexpected directions. We do feel that, like any good adventure, it is not the destination, but what we unexpectedly found around the bend that counts. Thus, the story we wish to tell here is the story of what, together with Enzo, we found around the bend: the unfailing virtues of the quantum tetrahedron.

¹a more pedantic rendering is: Virtue ignores the possibility of sliding down.

Our story starts by recalling that the (re)coupling theory of many $SU(2)$ angular momenta –framed mathematically in the structure of the Racah–Wigner tensor algebra– is the most exhaustive formalism in dealing with interacting many-angular momenta quantum systems [1, 2]. As such it has been over the years a common tool in advanced applications in atomic and molecular physics, nuclear physics as well as in mathematical physics. Suffices here to mention in physical chemistry the basic work of Wigner, Racah, Fano and others (see the collection of reprints [3] and the Racah memorial volume quoted in [7] below) as well as the recent book [4] on topics covered in this special issue.

In the last three decades there has been also a deep interest in applying (extensions of) such notions and techniques in the branch of theoretical physics known as Topological Quantum Field Theory, as well as in related discretized models for 3-dimensional quantum gravity. More recently the same techniques have been employed for establishing a new framework for quantum computing, the so-called “spin network” quantum simulator.

In previous work in collaboration with Enzo [5] we have stressed the combinatorial properties of Wigner $6j$ symbols (and of its generalizations, the $3nj$ symbols, see [6]) which stand at the basis of so many different fields of research.

The aim of the present paper is to discuss in details the apparent twofold nature of the $6j$ symbol displayed in quantum field theory and quantum computing, and to convey the idea that these two pictures actually merge together. In section 2 the $6j$ is looked at as a real “tetrahedron”, the basic *magic brick* in constructing 3-dimensional quantum geometries of the Regge type, while in section 3 it plays the role of a *magic box*, namely the elementary universal computational gate in a quantum circuit model. Thus the underlying physical models embody, at least in principle, the hardware of quantum computing machines, while a quantum computer of this sort, looked at as a universal, multi-purpose machine, might be able to simulate “efficiently” any other discrete quantum system. More remarks this topic are postponed to the end of section 3, while most mathematical definitions and results on Wigner $6j$ symbols needed in the previous sections are collected in Appendix A.

2 Tetrahedra and $6j$ symbols in quantum gravity

From a historical viewpoint the Ponzano–Regge asymptotic formula for the $6j$ symbol [7], reproduced in (14) of Appendix A.1, together with the seminal paper [8] in which “Regge Calculus” was founded, are no doubt at the basis of all “discretized” approaches to General Relativity, both at the classical and at the quantum level.

In Regge’s approach the edge lengths of a “triangulated” spacetime are taken as discrete counterparts of the metric, a tensorial quantity which encodes the dynamical degrees of freedom of the gravitational field and appears in the classical Einstein–Hilbert action for General Relativity through its second derivatives com-

bined in the Riemann scalar curvature. Technically speaking, a Regge spacetime is a piecewise linear (PL) "manifold" of dimension D dissected into *simplices*, namely triangles in $D = 2$, tetrahedra in $D = 3$, 4-simplices in $D = 4$ and so on. Inside each simplex either an Euclidean or a Minkowskian metric can be assigned: accordingly, PL manifolds obtained by gluing together D -dimensional simplices acquire an overall PL metric of Riemannian or Lorentzian signature².

Consider a particular triangulation $\mathcal{T}^D(\ell) \rightarrow \mathcal{M}^D$, where \mathcal{M}^D is a closed, locally Euclidean manifold of fixed topology and ℓ denotes collectively the (finite) set of edge lengths of the simplices in \mathcal{T}^D . The Regge action is given explicitly by (units are chosen such that the Newton constant G is equal to 1)

$$S(\mathcal{T}^D(\ell)) \equiv S^D(\ell) = \sum_{\sigma_i} \text{Vol}^{(D-2)}(\sigma_i) \epsilon_i, \quad (1)$$

where the sum is over $(D - 2)$ -dimensional simplices $\sigma_i \in \mathcal{T}^D$ (called hinges or "bones"), $\text{Vol}^{(D-2)}(\sigma_i)$ are their $(D - 2)$ -dimensional volumes expressed in terms of the edge lengths and ϵ_i represent the deficit angles at σ_i . The latter are defined, for each i , as $2\pi - \sum_k \theta_{i,k}$, where $\theta_{i,k}$ are the dihedral angles between pairs of $(D - 1)$ -simplices meeting at σ_i and labeled by some k . Thus a positive [negative or null] value of the deficit angle ϵ_i corresponds to a positive [negative or null] curvature to be assigned to the bone i , detected for instance by moving a D -vector along a closed path around the bone i and measuring the angle of rotation. Even such a sketchy description of Regge geometry should make it clear that a discretized spacetime is flat (zero curvature) inside each D -simplex, while curvature is concentrated at the bones which represent "singular" subspaces. It can be proven that the limit of the Regge action (1) when the edge lengths become smaller and smaller gives the usual Einstein–Hilbert action for a spacetime which is "smooth" everywhere, the curvature being distributed "continuously". Regge equations –the discretized analog of Einstein field equations– can be derived from the classical action by varying it with respect to the dynamical variables, *i.e.* the set $\{\ell\}$ of edge lengths of $\mathcal{T}^D(\ell)$, according to Hamilton principle of classical field theory (we refer to [9] for a bibliography and brief review on Regge Calculus from its beginning up to the 1990's).

Regge Calculus gave rise in the early 1980's to a novel approach to quantization of General Relativity known as Simplicial Quantum Gravity (see [9, 10, 11] and references therein). The quantization procedure most commonly adopted is the Euclidean path–sum approach, namely a discretized version of Feynman's path–

²Einstein's General Relativity corresponds to the physically significant case of a 4-dimensional spacetime endowed with a smooth Lorentzian metric. However, models formulated in "non-physical" dimensions such as $D = 2, 3$ turn out to be highly non trivial and very useful in a variety of applications, ranging from conformal field theories and associated statistical models in $D = 2$ to the study of geometric topology of 3-manifolds. Moreover, the most commonly used quantization procedure of such theories has a chance of being well-defined only when the underlying geometry is (locally) Euclidean, see further remarks below.

integral describing D -dimensional Regge geometries undergoing "quantum fluctuations" (in Wheeler's words a "sum over histories" [12], formalized for gravity in the so-called Hawking–Hartle prescription [13]). Without entering into technical details, the discretized path-sum approach turns out to be very useful in addressing a number of conceptual open questions in the approach relying on the geometry of smooth spacetimes, although the most significant improvements have been achieved for the $D = 3$ case, which we are going to address in some details in the rest of this section.

Coming to the interpretation of Ponzano–Regge asymptotic formula for the $6j$ symbol given in (14) of Appendix A.1, we realize that it represents the semiclassical functional, namely the semiclassical limit of a path-sum over all quantum fluctuations, to be associated with the simplest 3-dimensional "spacetime", an Euclidean tetrahedron T . In fact the argument in the exponential reproduces the Regge action $S^3(\ell)$ for T since in the present case $(D - 2)$ simplices are 1-dimensional (edges) and $\text{Vol}^{(D-2)}(\sigma_i)$ in (1) are looked at as the associated edge lengths, see the introductory part of Appendix A.

More in general, we denote by $\mathcal{T}^3(j) \rightarrow \mathcal{M}^3$ a particular triangulation of a closed 3-dimensional Regge manifold \mathcal{M}^3 (of fixed topology) obtained by assigning $SU(2)$ spin variables $\{j\}$ to the edges of \mathcal{T}^3 . The assignment must satisfy a number of conditions, better illustrated if we introduce the *state functional* associated with $\mathcal{T}^3(j)$, namely

$$\mathbf{Z}[\mathcal{T}^3(j) \rightarrow \mathcal{M}^3; L] = \Lambda(L)^{-N_0} \prod_{A=1}^{N_1} (-1)^{2j_A} w_A \prod_{B=1}^{N_3} \phi_B \left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\}_B \quad (2)$$

where N_0, N_1, N_3 are the number of vertices, edges and tetrahedra in $\mathcal{T}^3(j)$, $\Lambda(L) = 4L^3/3C$ (L is a fixed length and C an arbitrary constant), $w_A \doteq (2j_A + 1)$ are the dimensions of irreducible representations of $SU(2)$ which weigh the edges, $\phi_B = (-1)^{\sum_{p=1}^6 j_p}$ and $\{\:::\::\}_B$ are $6j$ symbols to be associated with the tetrahedra of the triangulation. Finally, the Ponzano–Regge *state sum* is obtained by summing over triangulations corresponding to all assignments of spin variables $\{j\}$ bounded by the cut-off L

$$\mathbf{Z}_{PR}[\mathcal{M}^3] = \lim_{L \rightarrow \infty} \sum_{\{j\} \leq L} \mathbf{Z}[\mathcal{T}^3(j) \rightarrow \mathcal{M}^3; L], \quad (3)$$

where the cut-off is formally removed by taking the limit in front of the sum.

It is not easy to review in short the huge number of implications and further improvements of Ponzano–Regge state sum functional (3), as well as its deep and somehow surprising relationships with so many different issues in modern theoretical physics and in pure mathematics. We are going to present in the rest of this section a limited number of items, whose selection is made mainly on the basis

of their relevance for (quantum) computational problems raised in the next section (we remind however the importance of this model in the so-called "loop" approach to quantum gravity [14], see also [11]).

- (a) As already noted in [7], the state sum $\mathbf{Z}_{PR}[\mathcal{M}^3]$ is a topological invariant of the manifold \mathcal{M}^3 , owing to the fact that its value is actually independent of the particular triangulation, namely does not change under suitable combinatorial transformations. Remarkably, these "moves" are expressed algebraically in terms of the relations given in Appendix A.2, namely the Biedenharn-Elliott identity (17) –representing the moves (2 *tetrahedra*) \leftrightarrow (3 *tetrahedra*)– and of both the Biedenharn-Elliott identity and the orthogonality conditions (18) for $6j$ symbols, which represent the barycentric move together its inverse, namely (1 *tetrahedra*) \leftrightarrow (4 *tetrahedra*).
- (b) In [15] a "regularized" version of (3) –based on representation theory of a quantum deformation of the group $SU(2)$ – was proposed and shown to be a well-defined *quantum invariant* for closed 3-manifolds³. Its expression reads

$$\mathbf{Z}_{TV}[\mathcal{M}^3; q] = \sum_{\{j\}} \mathbf{w}^{-N_0} \prod_{A=1}^{N_1} \mathbf{w}_A \prod_{B=1}^{N_3} \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix}_B, \quad (4)$$

where the summation is over all $\{j\}$ labeling highest weight irreducible representations of $SU(2)_q$ ($q = \exp\{2\pi i/r\}$, with $\{j = 0, 1/2, 1, \dots, r-1\}$), $\mathbf{w}_A \doteq (-1)^{2j_A} [2j_A + 1]_q$ where $[]_q$ denote a quantum integer, $\mathbf{w} = 2r/(q - q^{-1})^2$ and $| \vdots |_B$ represents here the q - $6j$ symbol whose entries are the angular momenta $j_i, i = 1, \dots, 6$ associated with tetrahedron B . If the deformation parameter q is set to 1 one gets $\mathbf{Z}_{TV}[\mathcal{M}^3; 1] = \mathbf{Z}_{PR}[\mathcal{M}^3]$.

It is worth noting that the q -Racah polynomial –associated with the q - $6j$ by a procedure that matches with what can be done in the $SU(2)$ case, see (16) in Appendix A.2– stands at the top of Askey's q -hierarchy collecting orthogonal q -polynomials of one discrete or continuous variable. On the other hand, the discovery of the Turaev-Viro invariant has provided major developments in the branch of mathematics known as geometric topology [16].

- (c) The Turaev-Viro or Ponzano-Regge state sums as defined above can be generalized in many directions. For instance, they can be extended to simplicial 3-manifold endowed with a 2-dimensional boundary [17] and to D -manifolds [18] (giving rise to topological invariants related to suitable (discretized) topological quantum field theory of the Schwarz type [19]).

³ The adjective "quantum" refers here to "deformations" of semi-simple Lie groups introduced by the Russian School of theoretical physics in the 1980's in connection with inverse scattering theory. From the mathematical viewpoint the Turaev-Viro invariant, unlike the Ponzano-Regge state sum functional, is always finite and has been evaluated explicitly for some classes of 3-manifolds.

- (d) The fact that the Turaev–Viro state sum is a topological invariant of the underlying (closed) 3–manifold reflects a crucial physical property of gravity in dimension 3 which makes it different from the corresponding $D = 4$ case. Loosely speaking, the gravitational field does not possess local degrees of freedom in $D = 3$, and thus any ”quantized” functional can depend only on global features of the manifold encoded into its overall topology. Actually the invariant (4) can be shown to be equal to the square of the modulus of the Witten–Reshetikhin–Turaev invariant, which in turn represents a quantum path–integral of an $SU(2)$ Chern–Simons topological field theory –whose classical action can be shown to be equivalent to Einstein–Hilbert action [20]– written for a closed oriented manifold \mathcal{M}^3 [21, 22]. Then there exists a correspondence

$$\mathbf{Z}_{TV}[\mathcal{M}^3; q] \longleftrightarrow |\mathbf{Z}_{WRT}[\mathcal{M}^3; k]|^2, \quad (5)$$

where the ”level” k of the Chern–Simons functional is related to the deformation parameter q of the quantum group.

Despite the ”topological” nature of Turaev–Viro (Ponzano–Regge) state sum and Witten–Reshetikhin–Turaev functionals in case of closed 3–manifolds, whenever a $2D$ –dimensional boundary occurs in \mathcal{M}^3 , giving rise to a pair (\mathcal{M}^3, Σ) , where Σ is an oriented surface (or possibly the disjoint union of a finite number of surfaces), things change radically. For instance, if we add a boundary to the manifold in Witten–Reshetikhin–Turaev quantum functional, the theory induced on Σ is a Wess–Zumino–Witten (WZW)–type Conformal Field Theory (CFT) [20], endowed with non–trivial quantum degrees of freedom. In particular, the frameworks outlined above can be exploit to establish a direct correspondence between $2D$ Regge triangulations and punctured Riemann surfaces, thus providing a novel characterization of the WZW model on triangulated surfaces on any genus [23] at a fixed level k .

We cannot enter here into many technical details on these developments. It should be sufficient to remark that, when addressing ”boundary” CFT, the geometric role of the quantum tetrahedron shades out, while its algebraic content is enhanced given that the (q) – $6j$ –symbol plays the role of a ”duality” (or ”fusion”) matrix, similar to a ”recoupling coefficient” between different basis sets, as (11) in Appendix A suggests.

- (e) In [24] a $(2 + 1)$ –dimensional decomposition of Euclidean gravity (which takes into account the correspondence (5)) is shown to be equivalent, under mild topological assumptions, to a Gaussian $2D$ fermionic system, whose partition function takes into account the underlying $3D$ topology. More precisely, the partition function for free fermions propagating along ”knotted loops” inside a 3–dimensional sphere corresponds to a $3D$ Ising model on so–called knot–graph lattices. On the other hand, the formal expression of $3D$ Ising partition function for a dimer covering of the underlying graph

lattice can be shown to coincide with the permanent of the generalized incidence matrix of the lattice [25, 26]. Recall first that the permanent of an $n \times n$ matrix A is given by

$$\text{per}[A] = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i, \sigma(i)} \quad (6)$$

where $a_{i, \sigma(i)}$ are minors of the matrix, $\sigma(i)$ is a permutation of the index $i = 1, 2, \dots, n$ and S_n is the symmetric group on n elements. A graph lattice \mathfrak{G} associated with a fixed orientable surfaces Σ of genus g embedded in S^3 may be constructed by resorting to the so-called "surgery link" presentation. Then the incidence matrix of such piecewise linear graph with, say, n vertices, is defined as an $n \times n$ matrix $A = (a_{ij})$ with entries in $(1, 0)$ according to whether vertices i, j are connected by an edge or not. Finally, the Ising partition function turns out to be a weighted sum –over all possible configurations of knot–graph lattices– of suitable "determinants" of generalized forms of the incidence matrices which take into account the topology of the underlying manifold. We skip however other technical details and refer to [27] for a short account of these results (which will be briefly reconsidered in the following section in the context of quantum computational questions).

The deep relationship between 3D quantum field theories that share a "topological" nature and (solvable) lattice models in 2D, sketched in the last item by resorting to a specific example, was indeed predicted in the pioneering paper by E. Witten [28]. Not so surprisingly, the basic quantum functional that realizes this connection was identified there with *the expectation value of a certain tetrahedral configuration of braided Wilson lines*, where "Wilson lines" are quantum observables associated with "particle trajectories" that in general look like sheafs of braided strands propagating from a surface Σ_1 to another Σ_2 , both embedded in a 3D background.

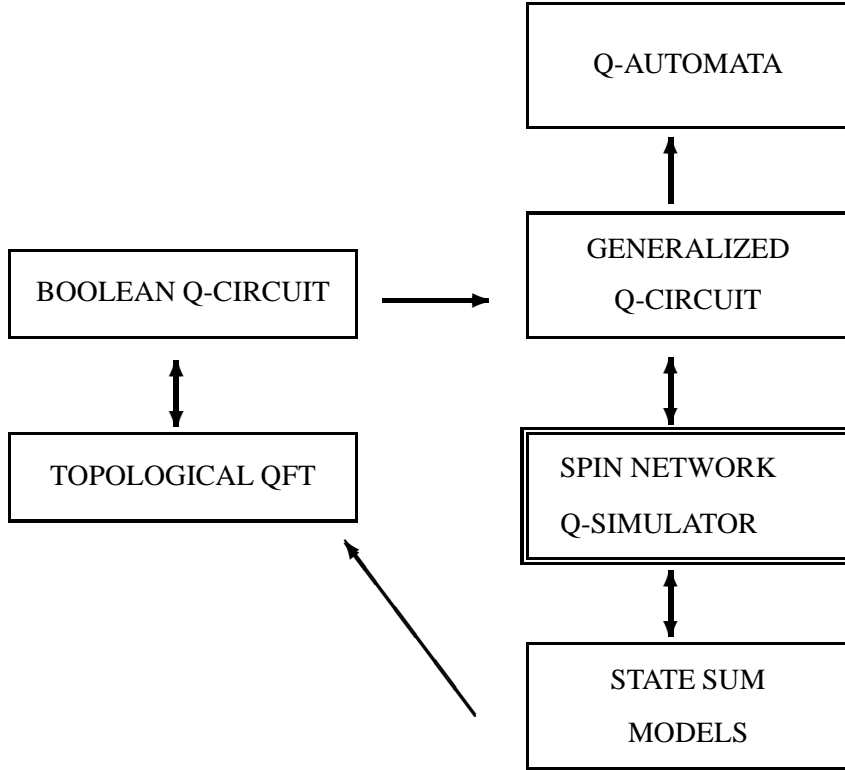
3 6j symbol and quantum algorithms

The model for universal quantum computation proposed in [29], the "spin network" simulator, is based on the (re)coupling theory of $SU(2)$ angular momenta as formulated in the basic texts [1, 2] on the quantum theory of angular momentum and the Racah–Wigner algebra respectively. At the first glance the spin network simulator can be thought of as a non–Boolean generalization of the Boolean *quantum circuit model*⁴ [30], with finite–dimensional, binary coupled computational

⁴ Recall that this scheme is the quantum version of the classical Boolean circuit in which strings of the basic binary alphabet $(0, 1)$ are replaced by collections of "qubits", namely quantum states in $(\mathbb{C}^2)^{\otimes N}$, and the gates are unitary transformations that can be expressed, similarly to what happens in the classical case, as suitable sequences of "elementary" gates associated with the Boolean logic operations *and*, *or*, *not*.

Hilbert spaces associated with N mutually commuting angular momentum operators and unitary gates expressed in terms of:

- i) recoupling coefficients ($3nj$ symbols) between inequivalent binary coupling schemes of $N = (n + 1)$ $SU(2)$ -angular momentum variables (j -gates);
- ii) Wigner rotations in the eigenspace of the total angular momentum \mathbf{J} (M -gates) (that however will not be taken into account in what follows, see section 3.2 of [29] for details)



In the diagram we try to summarize various aspects of the spin network simulator together with its relationships with other models for Q-computation, in the light of underlying physical frameworks discussed in the previous section.

On the left-hand portion of the diagram the standard Boolean quantum circuit is connected with a double arrow to the so-called topological approach to quantum computing developed in [31] (based, by the way, on the Witten-Reshetikhin-Turaev approach quoted in item (d) of the previous section). This means in practice that these two models of computation can be efficiently "converted" one into the other. The Boolean case is connected one-way to the box of the generalized Q-circuit because it is actually a particular case of the latter when all N angular momenta are $\frac{1}{2}$ -spins.

On the right-hand column, the double arrows stemming from the box of the spin

network Q-simulator relate it to its reference models: from the viewpoint of quantum information theory it is a generalized Q-circuit, as already noted before, while its physical setting can be assimilated to state sum-type models discussed in the first part of the previous section.

The upper arrow is to be meant as generating, from the general Q-computational scheme, families of "finite-states" Q-automata able to process in an efficient way a number of specific algorithmic problems that on a classical computer would require an exponential amount of resources (*cfr.* the end of this section).

Besides the features described above, the kinematical structure of the Q-spin network complies with all the requisites of an universal Q-simulator as defined by Feynman in [32], namely

- *locality*, reflected in the binary bracketing structure of the computational Hilbert spaces, which bears on the existence of poly-local, two-body interactions;
- *discreteness of the computational space*, reflected in the combinatorial structure of the (re)coupling theory of $SU(2)$ angular momenta [2, 33, 34];
- *discreteness of time*, given by the possibility of selecting controlled, step-by-step applications of sequences of unitary operations for the generation of (any) process of computation;
- *universality*, guaranteed by the property that any unitary transformation operating on binary coupled Hilbert spaces (given by $SU(2)$ $3nj$ symbols) can be reconstructed by taking a finite sequence of Racah-Wigner transforms implemented by expression of the type given in (11) of Appendix A (possibly apart from phases factors), as shown in [2], topic 12.

Then the Wigner $6j$ symbol plays a prominent role also in the spin network Q-simulator scheme, where it is the "elementary" unitary operation, from which any "algorithmic" procedure can be built up. The meaning of the identities (17) (18) satisfied by the $6j$'s in the present context is analyzed at length in [29], (section 4.2 and Appendix A) and can be related to the notion of intrinsic "parallelism" of quantum computers.

A caveat is however in order: the complexity class of any classical [quantum] algorithm is defined with respect to a "standard" classical [quantum] model of computation⁵. At the quantum level, such a reference model is the Boolean Q-circuit [30], and thus what is necessary to verify is that a $6j$ symbol with generic entries can be efficiently (polynomially) processed by a suitably designed Q-circuit. Note first that a $6j$ symbol with fixed entries, due to the finiteness of the Racah sum rule (see (16) in Appendix A.2), can be efficiently computed classically. On the other hand, the $6j$ is a $(2d+1) \times (2d+1)$ unitary matrix representing a change of basis,

⁵Recall that a quantum algorithm for solving a given computational problem is "efficient" if it belongs to the complexity class **BQP**, namely the class of problems that can be solved in polynomial time by a Boolean Q-circuit with a fixed bounded error in terms of the "size" of a typical input. In most examples the size of the input is measured by the length of the string of qubits necessary to encode the generic sample of the algorithmic problem, as happens with the binary representation of an integer number in calculations aimed to factorize it in prime factors.

as given explicitly in (11) of Appendix A, with j_{12}, j_{23} representing matrix indices running over an interval of length $2d + 1$ in integer steps. Thus the evaluation of the complexity class of this problem consists in asking whether, as d increases, the calculation of the $6j$ falls into the **BQP** class. The circuit which implements such task has been designed in [35] for the case of the $SU(2)_q$ $6j$ for each $q = \text{root of unity}$, while the analog problem involving the "classical", $SU(2)$ $6j$ is still open.

In the last few years two of the authors, in collaboration with S. Garnerone, have developed, on the basis of the spin network simulator setting [29], a new approach to deal with classes of algorithmic problems that classically admit only exponential time algorithms. The problems in questions arise in the physical context of $3D$ topological quantum field theories discussed in the previous section in the light of the fundamental result relating a topological invariant of knots, the Jones polynomial [36], with a quantum observable given by the vacuum expectation value of a Wilson "loop" operator [37] associated with closed knotted curves in the Witten–Reshetikhin–Turaev background model.

Without entering into technical details, efficient (polynomial time) quantum algorithms for approximating (with an error that can be made as small as desired) generalizations of Jones polynomial have been found in [35, 38], while the case of topological invariants of 3-manifolds has been addressed in [39]. The relevance in having solved this kind of problems stems from the fact that an approximation of the Jones polynomial is sufficient to simulate any polynomial quantum computation [40].

Summing up, the construction of such quantum algorithms actually bears on the interplay of three different contexts

1. a topological context, where the problem is well-posed and makes it possible to recast the initial instance from the topological language of knot theory to the algebraic language of braid group theory, as reviewed in [41];
2. a field theoretic context, where tools from $3D$ topological quantum field and associated $2D$ conformal field theory are used to provide a unitary representation of the braid group;
3. a quantum information context, where the basic features of quantum computation are used to efficiently solve the original problem formulated in a field theoretic language.

In the light of remark (e) at the end of section 2, further analysis of relationships between specific $3D$ topological quantum field theories and (solvable) lattice models in $2D$ in the quantum-computational context would represent a major improvement not only from a theoretical viewpoint, but also in view of possible physical implementations. In [27] some preliminary progress has been achieved for establishing a quantum algorithm for the evaluation of the permanent (6) associated with the partition function of the Ising model on knot-graph lattices. As shown in [42]

by resorting to numerical simulations, such a computational problem can be related to the computation of Jones invariants on suitably defined configurations, thus providing further evidence of the "universality" of any one of the quantum algorithms quoted above.

In conclusion, we hope to have been able to illustrate in sufficient details the role of the Wigner $6j$ symbol (or the $q-6j$) as an universal building block unifying such different fields as quantum geometry, topological quantum field theory, statistical lattice models and quantum computing.

The interplay between solvability and computability within the framework of quantum Witten–Reshetikhin–Turaev theory and solvable lattice models deserves however a few more comments. Unlike perturbatively renormalizable quantum field theory—which represent the basic tool in the standard model in particle physics, where the physically measurable quantities are obtained as finite limits of infinite series in the physical coupling constant—quantum WRT theory is actually "solvable" since functionals of type (5) and (4), as well as Wilson loop observables, are sums of a *finite number* of terms for each fixed value of the deformation parameter q . Actually such finiteness property reflects the existence of a deeper algebraic symmetry stemming from braid group representations and associated Yang–Baxter equation, see *e.g.* [37, 41] and references therein⁶. The issue of computability of all the relevant quantities of quantum WRT theory, and in particular of the Jones polynomial, is ultimately related to solvability/finiteness of the underlying theory. Thus the existence of "efficient" computational protocols should help in shedding light on the open question concerning the validation of the heuristic procedure associated with the path–sum quantization scheme (may be also in other contexts). Turning the argument upside down, the search for new efficient quantum algorithms for processing "invariant quantities" characterizing suitably decorated lattice, graphs, surfaces, *etc.* represents an original and possibly very fruitful approach for understanding the underlying physical models with respect to their (yet unknown) integrability properties.

Appendix A: the Wigner $6j$ symbol and its symmetries

Given three angular momentum operators $\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3$ —associated with three kinematically independent quantum systems—the Wigner–coupled Hilbert space of the composite system is an eigenstate of the total angular momentum

$$\mathbf{J}_1 + \mathbf{J}_2 + \mathbf{J}_3 \doteq \mathbf{J} \quad (7)$$

and of its projection J_z along the quantization axis. The degeneracy can be completely removed by considering binary coupling schemes such as $(\mathbf{J}_1 + \mathbf{J}_2) + \mathbf{J}_3$

⁶ This notion of solvability might be viewed as the quantum analog of the property of "complete integrability" in classical mechanics. Recall that integrable systems admit a sufficient number of conserved quantities that make it possible to solve explicitly Newton equations of motion. These "constants of motions" are endowed with a suitable algebraic structure under Poisson bracketing which is related in turn to complete integrability owing to Arnold–Liouville theorem.

and $\mathbf{J}_1 + (\mathbf{J}_2 + \mathbf{J}_3)$, and by introducing intermediate angular momentum operators defined by

$$(\mathbf{J}_1 + \mathbf{J}_2) = \mathbf{J}_{12}; \quad \mathbf{J}_{12} + \mathbf{J}_3 = \mathbf{J} \quad (8)$$

and

$$(\mathbf{J}_2 + \mathbf{J}_3) = \mathbf{J}_{23}; \quad \mathbf{J}_1 + \mathbf{J}_{23} = \mathbf{J}, \quad (9)$$

respectively. In Dirac notation the simultaneous eigenspaces of the two complete sets of commuting operators are spanned by basis vectors

$$|j_1 j_2 j_{12} j_3; jm\rangle \quad \text{and} \quad |j_1 j_2 j_3 j_{23}; jm\rangle, \quad (10)$$

where j_1, j_2, j_3 denote eigenvalues of the corresponding operators, j is the eigenvalue of \mathbf{J} and m is the total magnetic quantum number with range $-j \leq m \leq j$ in integer steps. Note that j_1, j_2, j_3 run over $\{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$ (labels of $SU(2)$ irreducible representations), while $|j_1 - j_2| \leq j_{12} \leq j_1 + j_2$ and $|j_2 - j_3| \leq j_{23} \leq j_2 + j_3$ (all quantum numbers are in \hbar units).

The Wigner $6j$ symbol expresses the transformation between the two schemes (8) and (9), namely

$$|j_1 j_2 j_{12} j_3; jm\rangle = \sum_{j_{23}} [(2j_{12} + 1)(2j_{23} + 1)]^{1/2} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix} |j_1 j_2 j_3 j_{23}; jm\rangle \quad (11)$$

apart from a phase factor⁷. It follows that the quantum mechanical probability

$$P = [(2j_{12} + 1)(2j_{23} + 1)] \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}^2 \quad (12)$$

represents the probability that a system prepared in a state of the coupling scheme (8), where j_1, j_2, j_3, j_{12}, j have definite magnitudes, will be measured to be in a state of the coupling scheme (9).

The $6j$ symbol may be written as sums of products of four Clebsch–Gordan coefficients or their symmetric counterparts, the Wigner $3j$ symbols. The relations between $6j$ and $3j$ symbols are given explicitly by (see *e.g.* [33])

$$\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix} = \sum (-)^{\Phi} \begin{pmatrix} a & b & c \\ \alpha & \beta & -\gamma \end{pmatrix} \begin{pmatrix} a & e & f \\ \alpha & \epsilon & -\varphi \end{pmatrix} \begin{pmatrix} d & b & f \\ -\delta & \beta & \varphi \end{pmatrix} \begin{pmatrix} d & e & c \\ \delta & -\epsilon & \gamma \end{pmatrix} \quad (13)$$

where $\Phi = d + e + f + \delta + \epsilon + \varphi$. Here Latin letters stand for j -type labels (integer or half-integers non-negative numbers) while Greek letters denote the associated magnetic quantum numbers (each varying in integer steps between $-j$ and j , $j \in \{a, b, c, d, e, f\}$). The sum is over all possible values of $\alpha, \beta, \gamma, \delta, \epsilon, \varphi$ with only three summation indices being independent.

⁷ Actually this expression should contain the Racah W -coefficient $W(j_1 j_2 j_3 j; j_{12} j_{23})$ which differs from the $6j$ by the factor $(-)^{j_1 + j_2 + j_3 + j}$. Recall that $(2j_{12} + 1)$ and $(2j_{23} + 1)$ are the dimensions of the representations labeled by j_{12} and j_{23} , respectively.

On the basis of the above decomposition it can be shown that the $6j$ symbol is invariant under any permutation of its columns or under interchange the upper and lower arguments in each of any two columns. These algebraic relations involve $3! \times 4 = 24$ different $6j$ with the same value and are referred to as *classical symmetries* as opposite to "Regge" symmetries to be discussed in A.2.

The $6j$ symbol is naturally endowed with a geometric symmetry, the *tetrahedral symmetry*, as the reproduction in Fig. 1 suggests. Note first that each $3j$ (or Clebsch–Gordan) coefficient vanishes unless its j -type entries satisfy the triangular condition, namely $|b - c| \leq a \leq b + c$, *etc.*. This suggests that each of the four $3j$'s in (13) can be associated with either a 3-valent vertex or a triangle. Accordingly, there are two graphical representation of the $6j$ exhibiting its symmetry properties. Here we adopt the three-dimensional picture introduced in the seminal paper by Ponzano and Regge [7], rather than Yutsis' "dual" representation as a complete graph on four vertices [34]. Then the $6j$ is thought of as a real solid tetrahedron T with edge lengths $\ell_1 = a + \frac{1}{2}, \ell_2 = b + \frac{1}{2}, \dots, \ell_6 = f + \frac{1}{2}$ in \hbar units⁸ and triangular faces associated with the triads $(abc), (aef), (dbf), (dec)$. This implies in particular that the quantities $q_1 = a + b + c, q_2 = a + e + f, q_3 = b + d + f, q_4 = c + d + e$ (sums of the edge lengths of each face), $p_1 = a + b + d + e, p_2 = a + c + d + f, p_3 = b + c + e + f$ are all integer with $p_h \geq q_k$ ($h = 1, 2, 3, k = 1, 2, 3, 4$). The conditions addressed so far are in general sufficient to guarantee the existence of a non-vanishing $6j$ symbol, but they are not enough to ensure the existence of a geometric tetrahedron T living in Euclidean 3-space with the given edges. More precisely, T exists in this sense if (*and only if*, see the discussion in the introduction of [7]) its square volume $V(T)^2 \equiv V^2$, evaluated by means of the Cayley–Menger determinant, is positive.

The features of the "quantum tetrahedron" outlined above represent the foundations of a variety of results, some of which were discovered in the golden age of quantum mechanics and have been widely used in old and present applications to atomic and molecular physics. In this paper we have tried to convey at least a few applications of this intriguing object in modern theoretical physics, while in the rest of this appendix we are going to complete the mathematical background needed in the previous sections, focusing in particular on semiclassical analysis and results from special function theory.

A.1 Ponzano–Regge asymptotic formula

The Ponzano–Regge asymptotic formula for the $6j$ symbol reads [7]

$$\left\{ \begin{matrix} a & b & d \\ c & f & e \end{matrix} \right\} \sim \frac{1}{\sqrt{24\pi V}} \exp \left\{ i \left(\sum_{r=1}^6 \ell_r \theta_r + \frac{\pi}{4} \right) \right\} \quad (14)$$

where the limit is taken for all entries $\gg 1$ (recall that $\hbar = 1$) and $\ell_r \equiv j_r + 1/2$

⁸ The $\frac{1}{2}$ -shift is shown to be crucial in the analysis developed in [7]: for high quantum numbers the length $[j(j+1)]^{1/2}$ of an angular momentum vector is closer to $j + \frac{1}{2}$ in the semiclassical limit.

with $\{j_r\} = \{a, b, c, d, e, f\}$. V is the Euclidean volume of the tetrahedron T and θ_r is the angle between the outer normals to the faces which share the edge ℓ_r .

From a quantum mechanical viewpoint, the above probability amplitude has the form of a semiclassical (wave) function since the factor $1/\sqrt{24\pi V}$ is slowly varying with respect to the spin variables while the exponential is a rapidly oscillating dynamical phase. Such kind of asymptotic behavior complies with Wigner's semiclassical estimate for the probability, namely $\left\{ \begin{smallmatrix} a & b & d \\ c & f & e \end{smallmatrix} \right\}^2 \sim 1/12\pi V$, to be compared with the quantum probability given in (12). Moreover, according to Feynman path sum interpretation of quantum mechanics [43], the argument of the exponential in (14) must represent a classical action, and indeed it can be read as $\sum p \dot{q}$ for pairs (p, q) of canonical variables (angular momenta and conjugate angles). Such an interpretation has been improved recently by resorting to multidimensional WKB theory for integrable systems and geometric quantization methods [44].

A.2 Racah hypergeometric polynomial

The generalized hypergeometric series, denoted by ${}_pF_q$, is defined on p real or complex numerator parameters a_1, a_2, \dots, a_p , q real or complex denominator parameters b_1, b_2, \dots, b_q and a single variable z by

$${}_pF_q \left(\begin{matrix} a_1 & \dots & a_p \\ b_1 & \dots & b_q \end{matrix} ; z \right) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}, \quad (15)$$

where $(a)_n = a(a+1)(a+2) \dots (a+n-1)$ denotes a rising factorial with $(a)_0 = 1$. If one of the numerator parameter is a negative integer, as actually happens in the following formula, the series terminates and the function is a polynomial in z .

The key expression for relating the $6j$ symbol to hypergeometric functions is given by the well-known Racah sum rule (see e.g. [2], topic 11 and [33], Ch. 9 also for the original references). The final form of the so-called *Racah polynomial* is written in terms of the ${}_4F_3$ hypergeometric function evaluated at $z = 1$ according to

$$\left\{ \begin{matrix} a & b & d \\ c & f & e \end{matrix} \right\} = \Delta(a, b, e) \Delta(c, d, e) \Delta(a, c, f) \Delta(b, d, f) (-)^{\beta_1} (\beta_1 + 1)! \\ \times \frac{{}_4F_3 \left(\begin{matrix} \alpha_1 - \beta_1 & \alpha_2 - \beta_1 & \alpha_3 - \beta_1 & \alpha_4 - \beta_1 \\ -\beta_1 - 1 & \beta_2 - \beta_1 + 1 & \beta_3 - \beta_1 + 1 & \end{matrix} ; 1 \right)}{(\beta_2 - \beta_1)! (\beta_3 - \beta_1)! (\beta_1 - \alpha_1)! (\beta_1 - \alpha_2)! (\beta_1 - \beta_3)! (\beta_1 - \alpha_4)!}, \quad (16)$$

where

$$\beta_1 = \min(a + b + c + d; a + d + e + f; b + c + e + f)$$

and the parameters β_2, β_3 are identified in either way with the pair remaining in the 3-tuple $(a + b + c + d; a + d + e + f; b + c + e + f)$ after deleting β_1 . The four α 's

may be identified with any permutation of $(a+b+e; c+d+e; a+c+f; b+d+f)$. Finally, the Δ -factors in front of ${}_4F_3$ are defined, for any triad (abc) as

$$\Delta(abc) = \left[\frac{(a+b-c)!(a-b+c)!(-a+b+c)!}{(a+b+c+1)!} \right]^{1/2}$$

Such a seemingly complicated notation is indeed the most convenient for the purpose of listing further interesting properties of the Wigner $6j$ symbol.

- The Racah polynomial is placed at the top of the Askey hierarchy including all of hypergeometric orthogonal polynomials of one (discrete or continuous) variable [45]. Most commonly encountered families of special functions in quantum mechanics are obtained from the Racah polynomial by applying suitable limiting procedures, as recently reviewed in [46]. Such an unified scheme provides in a straightforward way the algebraic *defining relations* of the Wigner $6j$ symbol viewed as an orthogonal polynomial of one discrete variable, *cfr.* (16). By resorting to standard notation from the quantum theory of angular momentum, the defining relations are:
the Biedenharn–Elliott identity ($R = a + b + c + d + e + f + p + q + r$):

$$\begin{aligned} \sum_x (-)^{R+x} (2x+1) \begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix} \begin{Bmatrix} c & d & x \\ e & f & q \end{Bmatrix} \begin{Bmatrix} e & f & x \\ b & a & r \end{Bmatrix} \\ = \begin{Bmatrix} p & q & r \\ e & a & d \end{Bmatrix} \begin{Bmatrix} p & q & r \\ f & b & c \end{Bmatrix}; \end{aligned} \quad (17)$$

the orthogonality relation (δ is the Kronecker delta)

$$\sum_x (2x+1) \begin{Bmatrix} a & b & x \\ c & d & p \end{Bmatrix} \begin{Bmatrix} c & d & x \\ a & b & q \end{Bmatrix} = \frac{\delta_{pq}}{(2p+1)}. \quad (18)$$

- Given the relation (16), the unexpected new symmetry of the $6j$ symbol discovered in 1958 by Regge [47] (see also [1, 33]) is recognized as a "trivial" set of permutations on the parameters α, β that leaves ${}_4F_3$ invariant. Combining the Regge symmetry and the "classical" ones, one get a total number of 144 algebraic symmetries for the $6j$. Note however that implications of Regge symmetry on the geometry of the quantum tetrahedron, taken into account in [48], certainly deserve further investigations also in view of the relevance of this topic in completely different contexts, *cfr.* for instance [49].
- The Askey hierarchy of orthogonal polynomials can be extended to a q -hierarchy [45], on the top of which the q - ${}_4F_3$ polynomial stands.
It is worth noting that the deformation parameter q was originally assumed by physicists to be a real number related to Planck constant \hbar by $q = e^{\hbar}$, and therefore it is commonly referred to as a 'quantum' deformation, while the 'classical', undeformed Lie group symmetry is recovered at the particular

value $q = 1$. However, when dealing with quantum invariants of knots and 3-manifolds formulated in the framework of "unitary" quantum field theory, as done in section 2 and 3, q is taken to be a complex root of unity, the case $q = 1$ being considered as the "trivial" one. We refer to [50, 51] for accounts on the theory of q -special functions and q -tensor algebras.

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